Quantum Mechanics as a Framework for Dealing with Uncertainty

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Abstract. Quantum uncertainty is described here in two guises: indeterminacy with its concomitant indeterminism of measurement outcomes, and fuzziness, or unsharpness. Both features were long seen as obstructions of experimental possibilities that were available in the realm of classical physics. The birth of quantum information science was due to the realization that such obstructions can be turned into powerful resources. Here we review how the utilization of quantum fuzziness makes room for a notion of approximate joint measurement of noncommuting observables. We also show how from a classical perspective quantum uncertainty is due to a limitation of measurability reflected in a fuzzy event structure – all quantum events are fundamentally unsharp.

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1. Introduction

Quantum physics arrived with a bang: the realization that experimental research had reached the limits of validity of classical physics. It was an understandable shock reaction of the pioneers of quantum mechanics to see this new theory as encapsulating obstructions – to the observation of atomic phenomena, to the application of the concepts of classical physics to atomic objects.

Thus, for example, the complementarity and uncertainty principles were commonly regarded as expressions of limitations: the former states the necessity of applying both the classical particle picture and the classical wave picture in the description of microsystems as well as the impossibility of the simultaneous application of these pictures; the latter was taken as quantifying this restriction of the simultaneous definition and measurement of canonically conjugate pairs of variables.

Another, related sort of limitation or obstruction was seen in the irreducibly probabilistic nature of quantum mechanics, which reflects the inherent indeterminacy of the values of observables and the fundamental indeterminism in the occurrence of measurement outcomes. This observation led directly to the quest for a crypto-deterministic, hidden variable description supposedly underlying quantum mechanics. The well-known no-hidden-variables theorems of von Neumann, Bell, Kochen and Specker, and others describe the extent to which any such description (as, for example, the de Broglie-Bohm theory) must differ from a classical-physical description in order to be consistent with quantum mechanics.

Entanglement is another fundamental quantum feature that was identified as leading to unpleasantly strange nonclassical behaviour – quantum nonlocality, as was noted in 1935 by Einstein, Podolsky and Rosen and by Schrödinger. Entanglement entails a limitation in the definition of the state of an individual object independently of its environment.

While quantum mechanics was enormously successfully applied in the analysis and exploitation of many new physical phenomena, it took many decades during which the above foundational issues were revisited and reviewed from new angles until finally the realization dawned that rather than being obstacles they are potential resources for information processing protocols which, if feasible, would be greatly more powerful in principle than methods based on classical physics. Hence we are witnessing a change of perspective from seeing quantum structures as *obstructions* to exploring them as *resources*.

In this contribution I will briefly review the notion of *quantum uncertainty* in its two guises as *indeterminacy* and *unsharpness*, and I will indicate how quantum uncertainty is fundamental to the interplay of quantum obstructions and novel quantum resources referred to above (Sec. 2). This is complemented with a sketch of a consistent and comprehensive way of presenting quantum mechanics as a classical fuzzy probability theory which makes precise the sense in which *from a classical perspective*, this theory reflects limitations of measurability. The existence of this particular classical

representation of quantum probability also demonstrates that the two forms of quantum uncertainty, indeterminacy and fuzziness, are in a way interchangeable and can be traded for each other (Sec. 3).

I dedicate this paper to Pekka Lahti on the occasion of his 60th birthday – we have spent many years trying together to understand aspects of the fundamental quantum features discussed here.

2. Quantum obstructions, or things to do with quantum uncertainty

2.1. Quantum uncertainty as indeterminacy

In a letter to Max Born dated December 4, 1926, Albert Einstein wrote these famous words:

> Quantum mechanics is very worthy of regard. But an inner voice tells me that this is not yet the right track. The theory yields much, but it hardly brings us closer to the Old ones secrets. I, in any case, am convinced that He does not play dice.

This is Einstein's rejection of the conclusion, suggested by Born's probabilistic interpretation of quantum mechanics [1], that the world should be fundamentally indeterministic. Instead of accepting *quantum uncertainty*, he initiates the search for *hidden variables*, a possibility hinted at by Born in his paper.

In classical probabilistic physical theories, all quantities are assigned sharp, definite values in every pure state. Quantum uncertainty in its first guise is the statement that within quantum mechanics, there is no state in which all observables would have definite values; and for every observable there are pure states (namely, superpositions of its eigenstates) in which their values are not definite, that is, *indeterminate*. If a state is understood as a probability catalogue for all measurement outcomes, then an observable would have a definite value in a given state if the probability distribution for the values of that observable is 0-1-valued. Gleason's theorem asserts that all states, defined as probability measures over the set of all closed subspaces of the complex Hilbert space (of dimension greater than 2) associated with a quantum system, are given by some density operator ρ via the trace formula:

$$prob(P) = trace(\rho\, P), \tag{1}$$

where P is the orthogonal projection onto a closed subspace and could figure as a spectral projection of a self-adjoint operator representing an observable. It is evident that quantum states are *not* 0-1-valued (or dispersion-free).

Gleason's theorem thus presents a severe obstacle for any attempt to supplement quantum mechanics with a classical description. Subsequent studies have shown that any hidden variable theory that reproduces all empirical predictions of quantum mechanics must incorporate some form of contextuality, that is, a dependence of the probability assignments on the measurement or preparation context. This is known to be true, in particular, for the best established hidden variable approach due to de Broglie and

Bohm. Rather than recalling the formal ingredients of a hidden variable theory in general, we will point these out in the construction of a classical embedding of quantum probability theory in the next section.

The Kochen-Specker and Bell theorems are refinements of Gleason's theorem, giving rise to an industry of attempts to specify minimally small sets of quantum propositions (projections) on which a consistent assignment of values 1 or 0 (true or false) cannot be defined (for an introduction and survey of this "coloring problem", see [2]). More recently, the converse problem has been solved, of describing large sets (of appropriate proposition structures) of projections that can be consistently assigned values 0 or 1. The result is laid down in the Bub-Clifton uniqueness theorem for interpretations of quantum mechanics [3]. Each possible rule of defining subsets of projections with definite values involves the choice of a quantum state and a reference observable, and can thus be taken as defining an interpretation of quantum mechanics by way of fixing the notion of reality of properties. The standard eigenstate-eigenvalue rule and Bohm's interpretation with a definite position variable are found to be examples of this general scheme. In this way one has learned to handle and quantify the limitation of ascribing truth values to quantum physical propositions.

Quantum indeterminacy is closely related the impossibility of distinguishing all pairs of distinct states through a single-shot measurement. In fact, suppose two different states ρ_1 and ρ_2 could be distinguished by the measurement of a single observable (represented in general by a positive operator measure, POM). There would thus be a pair of complementary outcomes, represented by positive operators (effects) E_1 and E_2 (such that $E_1 + E_2 = I$, the identity operator), such that ρ_1 will always trigger the outcome associated with E_1 and never that associated with E_2, and similarly ρ_2 will always lead to E_2 and never to E_1. It follows that the ranges of ρ_1, ρ_2 are mutually orthogonal subspaces (in fact, contained in the closed eigenspaces of E_1, E_2 associated with the eigenvalues 1, respectively. Hence nonorthogonal pairs of distinct states cannot be distinguished by a single measurement.

This impossibility, or quantum obstruction, has turned out to be fundamental to the security of quantum cryptographic protocols as it makes eavesdropping by way of single shot measurements impossible. It also makes superluminal signaling impossible.

A pure state of a compound system is *entangled* if it is not *separable*, i.e., if it is not a product state. This is equivalent to saying that there is no observable with nondegenerate eigenstates of product form whose values are definite. In this sense entanglement is an instance of quantum indeterminacy.

2.2. Quantum indeterminacy and indeterminism

The above Einstein quote alludes to the probabilistic nature of quantum mechanics, hence the *indeterminism* of this theory, which Einstein saw as a deficiency to be remedied in the course of future developments. We referred to the connection with the probability structure and the impossibility of defining noncontextual truth value assignments to all

experimental propositions of a quantum system, and arrived at the conclusion that according to quantum mechanics there is a fundamental *indeterminacy* of the values of most observables in any given state.

If the values of physical quantities are not definite, it would seem natural to conclude that a measurement of that quantity would not yield a predictable outcome. Indeed, it is commonly accepted that quantum mechanics only provides the probabilities for these outcomes. Hidden-variable theories are designed to restore the definiteness of (at least) some quantities, the values of which would then determine the outcomes of measurements. Thus determinateness should restore determinism. However, the necessary "hidden-ness" of these variables is just a representation of the fact that these hidden values cannot be accessed (measured or known) *as a matter of principle*. It seems a question of semantics whether or not one takes this observation as justification for the continued use of the term indeterminacy.

According to quantum mechanics, there is no empirically accessible cause for the occurrence of a particular measurement outcome if the state is not an eigenstate of the measured observable. The randomness of the outcomes is grounded in the fundamental indeterminateness and indeterminism that we call quantum uncertainty.

This uncertainty of measurement outcomes for quantum objects is being exploited in the theory of *quantum games* which often show peculiar advantages for a party using quantum strategies compared to classical games.

2.3. Quantum uncertainty as fuzziness

Nonorthogonal pure states cannot be distinguished by a single-shot measurement. Geometrically, such states belong to different orthonormal bases, and as such they are eigenstates of *noncommuting observables*. Thus we see that quantum uncertainty is closely related to the existence of incompatible pairs of observables. The above proof of the indistinguishability of nonorthogonal states makes implicit use of the fact that the two observables of which these states are eigenstates cannot be measured together.

The Heisenberg uncertainty principle comprises three physical statements which actually have been proven as theorems in quantum mechanics for certain pairs of observables; we phrase them here informally:

(a) the values of two noncommuting quantities can be *unsharply defined* to the extent allowed by the uncertainty relation for the widths of their distribution in the given state;

(b) the values of two noncommuting quantities can be *jointly and approximately measured* to accuracies allowed by a measurement uncertainty relation;

(c) the initially sharp value of a quantity A will be disturbed (made unsharp) by a subsequent measurement of a noncommuting quantity B such that the inaccuracy of the B-measurement and the magnitude of the disturbance of A obey an uncertainty relation.

These are the state preparation, joint measurement, and disturbance versions of the uncertainty relation. The last version is an expression of the *Heisenberg effect*.

Note that the above formulations state *positive* possibilities, in contrast to the traditional way of phrasing the uncertainty principle as a *limitation* of preparations or measurements. Statement (a) is well known and universally accepted whereas (b) and (c) have remained contentious for many decades, due to the lack of a rigorous formulation. These latter two statements have been made precise only rather recently on the basis of the generalised representation of observables as POMs, which allowed the introduction of operationally relevant concepts of approximate (joint) measurements and of suitable measures of inaccuracy.

2.3.1. Joint measurability. Two observables, represented as POMs M_1, M_2 on the real line $\mathbb{R}$ (say), are jointly measurable if they are marginals of a third observable M defined on $\mathbb{R}^2$; this means that for all (Borel) subsets X, Y of $\mathbb{R}$, one has $M_1(X) = M(X \times \mathbb{R})$ and $M_2(Y) = M(\mathbb{R} \times Y)$. This notion captures the idea that if two quantities can be measured together they must have a joint probability distribution for every state.

It is well known that two *sharp observables* (represented by spectral measures) are jointly measurable if and only if they commute, but two noncommuting *unsharp observables* (POMs that are not projection valued) can be jointly measurable. Unsharpness is thus a prerequisite for joint measurability.

There are not many general results on the joint measurability of pairs of noncommuting observables. Early positive observations after von Neumann's and Wigner's no-go theorems (the latter based on the failure of the Wigner function to be nonnegative) were the discovery of the Husimi function, which was much later understood to give rise to an instance of a covariant phase space observables. It became gradually clear that suitable smeared versions of position and momentum are indeed jointly measurable, and that in all instances an inaccuracy tradeoff relation is satisfied. This led to a notion of unsharp observables and approximate (joint) measurements which can be expected to provide the basis for a general theory of joint measurability in quantum mechanics (Fig. 1).

The question of the approximate joint measurability of position and momentum observables and that of pairs of qubit observables have been analysed in great generality and are now well understood. The solution to the joint measurement problem for position-momentum case was obtained in a breakthrough paper by Werner [4]. It is now known that a trade-off relation

$$d(Q, M_1)\, d(P, M_2) \geq C\hbar \qquad (C > 0) \tag{2}$$

must hold for measures of the distances $d(Q, M_1)$ and $d(P, M_2)$ between position and momentum on the one hand and a pair of approximating observables (POMs) M_1 and M_2 on the other hand if the latter are to be jointly measurable. This is, finally, Heisenberg's famous though debated *joint measurement inaccuracy relation* made rigorous in operationally meaningful terms. Alternative formulations of (2) based

$$
\begin{tikzcd}
 & G(X\times Y) \arrow[dl, "{[Y=\mathbb{R}]}"'] \arrow[dr, "{[X=\mathbb{R}]}"] & \\
G_1(X) \arrow[d, squiggly] & & G_2(Y) \arrow[d, squiggly] \\
E(X) & & F(Y)
\end{tikzcd}
$$

Figure 1. Scheme of an approximate joint measurement of two (sharp or unsharp, noncommuting) observables E, F with values in $\mathbb{R}$. The measured observable G has values in $\mathbb{R}^2$, its Cartesian marginals G_1, G_2 are approximators to E, F, respectively. The quality of approximation can be quantified by means of suitable measures of (in)accuracy.

on a measure of *error bar widths* have been found subsequently [5]. It turns out that finite Werner distance implies finite error bar widths and that there are more jointly measurable pairs of observables M_1, M_2 approximating Q, P in the sense of finite error bar widths than there are with finite Werner distances. For a recent review of Heisenberg's uncertainty principle exemplified for position and momentum, see [6].

Approximate joint measurements of qubit observables were studied in [7] and the joint measurability of a pair of simple qubit observables was characterised independently in [8, 9, 10]. We recall this latter result briefly, starting with the example of a POM that represents a joint measurement of *all* spin directions of a spin-$\frac{1}{2}$ system.

Let S^2 denote the unit sphere, with its σ-algebra of Borel sets $\mathcal{B}(S^2)$ and the rotationally invariant measure $d\Omega(\mathbf{n})$, normalized as $\Omega(S^2) = 4\pi$. (Here $\mathbf{n}$ denotes a unit vector labeling a point on S^2.) Then the following defines a normalized POM:

$$\mathcal{B}(S^2) \ni Z \mapsto G(Z) := \frac{1}{2\pi}\int_Z \frac{1}{2}(I + \mathbf{n}\cdot\boldsymbol{\sigma})\, d\Omega(\mathbf{n})\,. \tag{3}$$

Now, if $Z(\pm\mathbf{n}_o)$ denotes the hemisphere with centre $\pm\mathbf{n}_o$, we obtain:

$$G\left(Z(\pm\mathbf{n}_o)\right) = \frac{1}{2}(I \pm \frac{1}{2}\mathbf{n}_o\cdot\boldsymbol{\sigma})\,. \tag{4}$$

Note that $G\left(Z(\mathbf{n}_o)\right) + G\left(Z(-\mathbf{n}_o)\right) = I$, so that these two positive operators constitute a 2-valued (so-called simple) observable. Each of these observables are smeared or fuzzy versions of the associated sharp observables defined by the projections $P(\pm\mathbf{n}_o) := \frac{1}{2}(I \pm \mathbf{n}_o\cdot\boldsymbol{\sigma})$, in the sense that one can write $\frac{1}{2}(I + \frac{1}{2}\mathbf{n}_o\cdot\boldsymbol{\sigma}) = \frac{3}{4}P(\mathbf{n}_o) + \frac{1}{4}P(-\mathbf{n}_o)$ and similarly for the other effect. One can also say that these smeared versions are approximations to the corresponding sharp observables. Thus the observable G represents an approximate joint measurement of all sharp spin projections.

In general, a qubit effect can be represented as an operator $A = a_0 I + \mathbf{a}\cdot\boldsymbol{\sigma}$, with the constraint that the eigenvalues lie in $[0,1]$, hence $0 \le a_0 \pm |\mathbf{a}| \le 1$, or:

$$|\mathbf{a}| \le \min(a_0, 1 - a_0). \tag{5}$$

The result mentioned above now reads as follows. Two qubit effects $A = a_0 I + \mathbf{a}\cdot\boldsymbol{\sigma}$, $B = b_0 I + \mathbf{b}\cdot\boldsymbol{\sigma}$ are jointly measurable if and only if they satisfy the inequality

$$\frac{1}{2}[\mathcal{F}(2-\mathcal{B}) + \mathcal{B}(2-\mathcal{F})] + (xy - 4\mathbf{a}\cdot\mathbf{b})^2 \geq 1. \tag{6}$$

Here the following abbreviations are used:

$$\begin{aligned}
\mathcal{F} \quad &:= \varphi(A)^2 + \varphi(B)^2;\\
\mathcal{B} \quad &:= \beta(A)^2 + \beta(B)^2;\\
x \quad &:= \varphi(A)\beta(A) = 2a_0 - 1;\\
y \quad &:= \varphi(B)\beta(B) = 2b_0 - 1;\\
\varphi(A) &:= \sqrt{a_0^2 - |\mathbf{a}|^2} + \sqrt{(1-a_0)^2 - |\mathbf{a}|^2};\\
\beta(A) &:= \sqrt{a_0^2 - |\mathbf{a}|^2} - \sqrt{(1-a_0)^2 - |\mathbf{a}|^2}.
\end{aligned}$$

The quantities $\varphi(A)$ and $\beta(A)$ are measures of *unsharpness* (or fuzziness) and *bias*, respectively [8, 11]. In the unbiased case, where $a_0 = b_0 = \frac{1}{2}$, Eq. (6) assumes the much simplified form

$$16|\mathbf{a}\times\mathbf{b}|^2 \leq (1 - 4|\mathbf{a}|^2)(1 - 4|\mathbf{b}|^2)\,, \tag{7}$$

or equivalently,

$$|\mathbf{a}+\mathbf{b}| + |\mathbf{a}-\mathbf{b}| \leq 1\,. \tag{8}$$

The two factors on the right hand side of the former inequality are also measures of the unsharpness of the two effects A, B and are thus seen to be constrained by the noncommutativity of A, B (as quantified by the vector product term on the left hand side, which is proportional to $\|[A,B]\|^2$).

With the choices $\mathbf{a} = \frac{1}{4}\mathbf{n}_o$, $\mathbf{b} = \frac{1}{4}\mathbf{n}'_o$, we reproduce the effects $G\left(Z(\mathbf{n}_o)\right)$, $G\left(Z(\mathbf{n}'_o)\right)$, and it is easily seen that the last two inequalities are satisfied in this case. This inequality describes the degree of unsharpness in the two noncommuting effects required for them to be jointly measurable.

We conclude that the strict obstruction to the joint measurability of noncommuting quantities can be lifted if a sufficient degree of fuzziness is allowed in the definition of the observables and their measurements. This step opens the door to the introduction of *informationally complete* observables, whose statistics allow the complete identification of all states and thus give rise to realizations of quantum state tomography protocols.

This transformation of quantum uncertainty from vice to virtue was a move envisaged by Heisenberg in 1927 who, however, lacked the formal tools to make precise the corresponding measurement versions of his uncertainty principle. In his famous Como lecture [12], Bohr endorsed this positive outlook as follows:

> In the language of the relativity theory, the content of the relations (2) [the uncertainty relations] may be summarized in the statement that according to the quantum theory a general reciprocal relation exists between the maximum sharpness of definition of the space-time and energy-momentum

> vectors associated with the individuals. This circumstance may be regarded as a simple symbolical expression for the complementary nature of the space-time description and claims of causality. At the same time, however, the general character of this relation makes it possible to a certain extent to reconcile the conservation laws with the space-time co-ordination of observations, the idea of a coincidence of well-defined events in a space-time point being replaced by that of unsharply defined individuals within finite space-time regions.

Incidentally, this passage appears to be the first occurrence of the word *unsharp* in the quantum physics literature; hence it would seem that this teutonic extension to the English vocabulary is due to Bohr.

2.3.2. Heisenberg effect. Having introduced the idea of the approximation of one observable by another observable, one can use the associated measures of distance or inaccuracy to quantify the inevitable disturbance of a quantum state through a measurement. If the system is in an eigenstate of the measured quantity, then a Lüders measurement of that quantity does not alter the state (see, e.g., [13]). However, consider a measurement of position on a state in which the momentum is fairly well defined (as represented by a rather sharply peaked distribution). If the position measurement is of good quality, the momentum distribution afterwards will no longer be sharply peaked.

This intuitive consideration can be made precise: if the position measurement is followed by a momentum measurement, one will be able to see the disturbance of the momentum through the position measurement: the final momentum measurement statistics can be expressed in terms of the state prior to the interfering position measurement, and this leads to the definition of a POM M that would be the sharp momentum observable if no position measurement had been made. If the position measurement is sharp, then it can be shown that the observable M must actually commute with the position observable and thus gives no information at all about the momentum distribution of the initial state (Fig. 2(a))! If instead of sharp position an approximate measurement of position is made in an appropriate way, then the subsequent momentum measurement defines an unsharp momentum observable M relative to the initial state (Fig. 2(b)). In fact, the whole sequence of both measurements constitutes a joint approximate measurement of position and momentum relative to the initial state. (Details of the proof are reviewed in [6].) The degree of inaccuracy of the resulting approximate momentum measurement represented by M is a measure of the disturbance of the original, undisturbed (by the position measurement) momentum distribution, and the inaccuracy inequality (2), which holds for this joint (sequential) measurement, describes the tradeoff between the position measurement inaccuracy and the disturbance of momentum, thus confirming version (c) of the uncertainty principle.

The Heisenberg effect – the necessary disturbance of the quantum state through measurement, can thus be quantified by suitable joint measurement inaccuracy relations. If an eavesdropper observes a quantum communication channel, her measurement will disturb the transmitted qubit unless the state is an eigenstate of the chosen

$$(a)\qquad \begin{tikzcd} \rho \arrow[r] \arrow[d] & \mathsf{Q} \arrow[r] \arrow[d] & \rho' \arrow[r] & \mathsf{P} \arrow[d] \\ \rho^{\mathsf{Q}}, \rho^{\mathsf{P}} & \rho^{\mathsf{Q}} & & \rho'^{\mathsf{P}} = \rho^{f(\mathsf{Q})} =: \rho^{G_2} \end{tikzcd}$$

$$(b)\qquad \begin{tikzcd} \rho \arrow[r] \arrow[d] & G_1 \arrow[r] \arrow[d, squiggly] & \rho'' \arrow[r] & \mathsf{P} \arrow[d] \\ \rho^{\mathsf{Q}}, \rho^{\mathsf{P}} & \rho^{\mathsf{Q}} & & \rho''^{\mathsf{P}} =: \rho^{G_2} \end{tikzcd}$$

Figure 2. Momentum disturbance through position measurement. (a) A (nonselective) measurement of sharp position, $G_1 := \mathsf{Q}$, leads to a state ρ', on which a measurement of sharp momentum is performed. This defines a POM G_2 relative to ρ whose effects are functions of position and thus G_2 is not even an approximation to the sharp momentum P. (b) A suitable form of measurement of an approximate position observable G_1 is performed, leaving the system in a state ρ'', on which a measurement of sharp momentum is carried out. This defines a POM G_2 relative to ρ which is an approximation to P.

measurement, but this is beyond the control of the eavesdropper and thus the measurement leaves a detectable trace. In this way the Heisenberg effect becomes decisive for the security of cryptographic protocols.

We conclude that the noncommutativity of pairs of quantum observables necessitates the introduction of a degree of fuzziness in order to enable approximate joint measurability of such pairs.

3. Quantum probability as classical fuzzy probability

In quantum mechanics the distinction between sharp and unsharp observables is operationally meaningful [14]. Still, there is a fundamental degree of fuzziness even in the case of sharp quantum observables, represented by projection valued measures. Consider, for example, two rank-1 projections P_1 and P_2 that are neither orthogonal nor identical. In the pure state $\rho = P_1$, the probability for P_2 is neither 1 nor 0. By contrast, in the corresponding classical situation this latter probability would be 0. We now show that from a classical perspective, *all* effects of a quantum system are fuzzy properties, whether they are projections or just non-idempotent positive operators.

Let $\mathcal{S}_q$ and $\mathcal{S}_c$ denote the convex sets of (probabilistic) states of a quantum or classical system, respectively, with $\mathcal{E}_q$ and $\mathcal{E}_c$ being the associated sets of effects. Thus $\mathcal{S}_q$ is the set of all density operators, $\mathcal{E}_q$ the set of all positive operators E such that $O \leq E \leq I$. Each effect defines a unique affine functional on the set of states, which is

given by the trace formula,

$$E[\rho] := \mathrm{tr}[\rho E] \equiv p_\rho(E).$$

This is the probability that the measurement of E give a positive outcome indicating the occurrence of the event associated with E in state ρ.

We can think of $\mathcal{S}_c$ as (a dense convex subset of) the convex set of all probability probability measures on a measurable space (Ω, Σ) and $\mathcal{E}_c$ the set of all measurable functions $f : \Omega \to \mathbb{R}$ with values between 0 and 1. Each classical effect is thus a fuzzy or crisp set and defines an affine functional on the set of probability measures via

$$f[\mu] := \int_\Omega f d\mu \equiv p_\mu(f),$$

giving the probability of the event associated with effect f in state μ.

To approach the formalization of a hidden-variable description of quantum mechanics, we note that there are two canonical ways of relating the quantum statistical model $(\mathcal{S}_q, \mathcal{E}_q)$ with a classical statistical model $(\mathcal{S}_c, \mathcal{E}_c)$.

3.1. Classical embedding

Let $\Phi : \mathcal{S}_q \to \mathcal{S}_c$ be an affine mapping (i.e. a mapping that preserves convex combinations). This fixes a "dual" mapping $\Phi' : \mathcal{E}_c \to \mathcal{E}_q$ via

$$p_\rho(\phi'(f)) \equiv \Phi'(f)[\rho] = f[\Phi(\rho)] \equiv p_{\phi(\rho)}(f)$$

for all $\rho \in \mathcal{S}_q$, $f \in \mathcal{E}_c$. The map Φ' is interpreted as a *quantization map*. If one wants to ensure that all quantum effects are covered, that is, Φ' is surjective, it follows that Φ must be injective. There is a unique family of solutions which is easily characterized: each such injective affine map Φ is generated by (and conversely defines) a unique informationally complete observable $A : \Sigma \to \mathcal{E}_q$ via $\Phi(\rho) = p^A_\rho$, where

$$p^A_\rho(X) = \mathrm{tr}[\rho A(X)], \quad X \in \Sigma.$$

Then every quantum state is uniquely associated with a probability measure; but the desired surjectivity of Φ' has not been achieved: it can be shown not every quantum effect is an image of a classical effect. Only a suitably defined (unique) linear extension of Φ' to all bounded measurable functions is surjective. This means that some quantum effects are represented by functions f that are not nonnegative. This deficiency is reminiscent of the "dual" deficiency of the Wigner function representation of quantum states.

We conclude that classical embeddings via informationally complete observables give rise only to *partial* classical representations of a quantum statistical model. Still, this representation is almost complete in a formal sense as all the effects in the range of the observable A are in the range of Φ' (in fact, $A(X) = \Phi'(1_X)$, where 1_X denotes the indicator function of the set X), and their span is dense in the space of selfadjoint bounded operators.

3.2. Classical extensions

The only way of defining a comprehensive classical representation of a quantum statistical model via an affine correspondence is through a *reduction map* $\Psi : \mathcal{S}_c \to \mathcal{S}_q$ and its associated dual map $\Psi' : \mathcal{E}_q \to \mathcal{E}_c$. Since one wants coverage of all quantum states, Ψ is required to be surjective. A solution was introduced by Misra [15] and cast in the framework of quantum and classical statistical dualities by Bugajski [16, 17]. (A detailed review of the literature on this subject is found in [18].)

Let now $\Omega_q := \mathcal{S}_q^{pure}$ be the set of pure quantum states, represented as rank-1 projections, equipped with its natural algebra Σ of Borel subsets. (For a qubit this is the surface of the Bloch sphere.) Let δ_ω denote the point (Dirac) measure concentrated on $\omega \in \Omega_q$. Then we define:

$$\mathcal{S}_c \ni \mu = \int_{\Omega_q} \delta_\omega \, d\mu(\omega) \mapsto \Psi_M(\mu) := \int_{\Omega_q} \omega \, d\mu(\omega) \equiv \rho_\mu \in \mathcal{S}_q \tag{9}$$

Since every convex decomposition of a density operator ρ can be cast in the form of such an integral, it is clear that this affine map Ψ_M is surjective. Noting that

$$\mathrm{tr}[\rho_\mu E] = \int_{\Omega_q} \mathrm{tr}[\omega E] d\mu(\omega) = \int_{\Omega_q} f_E(\omega) \, d\mu(\omega)$$

we obtain the dual correspondence

$$\mathcal{E}_q \ni E \mapsto \Psi'_M(E) = f_E \in \mathcal{E}_c, \qquad f_E(\omega) = \mathrm{tr}[\omega E].$$

Here it is seen that every quantum effect E – whether sharp (i.e., projection) or unsharp – is represented classically as a fuzzy set f_E. The set of all classical effects of this form is a subset of the full set of classical effect and not sufficient to separate all classical states.

The classical representation of quantum mechanics through the above map Ψ_M was conceived rather intuitively (and in fact formulated rigorously) in the work of Misra. This leaves open the question whether there are any alternative constructions. It has been shown recently [18] that the map Ψ_M gives the essentially unique non-redundant representation in the following sense. Nonredundance means that one starts with a phase space Ω and requires that the reduction map Ψ is such that there is a correspondence ι between Ω and the set of pure quantum states Ω_q, $\iota(\omega) = \Psi(\delta_\omega)$. As a reduction map, Ψ also has to satisfy a certain physically natural continuity property. It then follows that Ψ can be represented according to

$$\mathrm{tr}[\Psi(\mu)E] = \int_{\Omega} \mathrm{tr}[\iota(\omega)E]\mu(d\omega) = \int_{\Omega_q} \mathrm{tr}[\omega' E](\mu \circ \iota^{-1})(d\omega'), \tag{10}$$

where $\mu \in \mathcal{S}_c$ and $E \in \mathcal{E}_q$. This generalized representation Ψ differs from the Misra map Ψ_M by the map ι, which essentially effects a relabeling of the set of pure quantum states ($\Psi(\mu) = \Psi_M(\mu \circ \iota^{-1})$). The dual map is $\Psi'(E) = f_E$, with $f_E(\omega) = \mathrm{tr}[\iota(\omega)E]$. This shows that also in this generalized case all quantum effects are represented as classical fuzzy sets. Examples of maps Ψ with different ι are worked out in [18].

4. Conclusion

We have reviewed the notion of quantum uncertainty in its two forms of indeterminacy and fuzziness, or unsharpness, and we have shown how quantum uncertainty underlies many of the infamous quantum restrictions that have recently been rediscovered as resources for information processing. We showed how quantum fuzziness gives room for the approximate joint measurability of noncommuting observables, which is necessary for informational completeness and quantum state tomography.

We also have reviewed the only "good" classical representation of the quantum statistical model $(\mathcal{S}_q, \mathcal{E}_q)$, and found that it confirms the known no-go theorems for hidden variable supplementations of quantum mechanics: this representation preserves quantum uncertainty in the form of a fundamental fuzziness. In fact, the distinction between sharp and unsharp quantum effects has been blurred as all quantum effects are represented as fuzzy classical effects. Even though the classical statistical model contains all Dirac measures, that is, the pure states, classical events on which they are dispersion free, the sharp classical properties, do not occur in the representation. One may say that indeterminacy and fuzziness have become indistinguishable.

From this classical perspective, quantum uncertainty is due to a limitation of measurability: only effects of the form f_E are measurable, and their measurement does not allow one to distinguish the different possible convex decomposition of a mixed quantum state. Nor is it possible in a single shot measurement to pinpoint a pure state.

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